Math 261B Thurs. 10/29

$e_i \mapsto \sum_j a_{ji} e_j$

Examples Chevalley bases in V^λ

(① GL_n $V^\lambda = V^{(1^d, 0, \ldots, 0)} = \Lambda^d K^n$:

Wedge monomials $e_{i_1} \wedge \cdots \wedge e_{i_d}$ form a Chevalley basis)

$\langle e_I \rangle$ A e_J

$=$ the $d \times d$ minor of A in rows I, cols J.

E_i F_i $F_i^{(m)} / m!$

$E_i^{(m)} / m!$

② GL_n $V^\lambda = V^{(d, 0, \ldots, 0)} = S^d K^n = K[x_1, \ldots, x_n]_d$

$v_\lambda = x_1^d$ E_i $x_i \frac{\partial}{\partial x_{i+1}}$ F_i $x_{i+1} \partial / \partial x_i$

$F_i^{(m)} = \frac{x_{i+1}^m}{m!} (\partial / \partial x_i)^m$

$\left(\frac{x^m}{m!} \right)' = \frac{x^{m-1}}{(m-1)!}$

$F_i^{(m)} \frac{x_i^k}{k!} = \frac{x_{i+1}^m}{m!} \frac{x_i^{k-m}}{(k-m)!}$

$\frac{(\ell+m)!}{\ell! \, m!}$

$F_i^{(m)} \frac{x_i^k}{k!} \frac{x_{i+1}^\ell}{\ell!} = \binom{\ell+m}{\ell} \frac{x_{i+1}^{\ell+m}}{(\ell+m)!} \frac{x_i^{k-m}}{(k-m)!}$

Basis $\left\{ \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_n^{k_n}}{k_n!} \right\}_{k_1+\cdots+k_n=d}$ $\qquad \frac{x_1^d}{d!}$

is a " $\longleftarrow$ $F_1^{(k_2+\cdots+k_n)} F_2^{(k_3+\cdots+k_n)} \cdots x_1^d/d!$

Chevalley basis for $V^\lambda_{\mathbb{Z}}$

$$\bigoplus_d V_{\mathbb{Z}}^{(d,0,\dots,0)} = D(x_1,\dots,x_n) \text{ divided power algebra}$$

(alternatively: $\binom{d}{k_1,\dots,k_n} x_1^{k_1} \cdots x_n^{k_n}$ is the Chevalley basis

with $v_\lambda = x_1^d$)

$$\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj}$$

$$\mathcal{O}_{\mathbb{Z}}(Gl_n) = \mathbb{Z}[a_{11},\dots,a_{nn}, \det(A)^{-1}]$$

$V^{(1,\dots,1)} = \det$

$V^{(-1,\dots,-1)} = (\det)^{-1}$

Linear substitution $\qquad V^{(1,0,\dots,0)} = K^n \qquad \sum_k b_{ik} c_{kj}$

$$(x_1,\dots,x_n) \mapsto (x_1,\dots,x_n)A$$

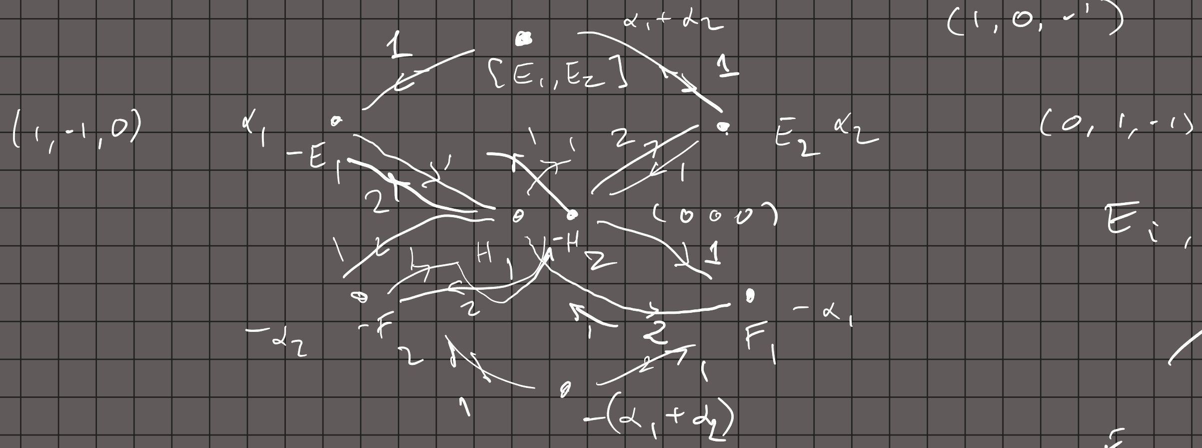
$\begin{cases} V_{\mathbb{Z}}^{(1,0,\dots,0)} & a_{ij} \\ V_{\mathbb{Z}}^{(-1,\dots,-1)} & \det(A)^{-1} \end{cases}$

$$\frac{x_j^k}{k!} \longmapsto \Big(\sum_i a_{ij} x_i\Big)^k / k!$$

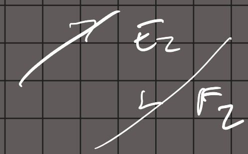
$$= \sum_{k_1+\cdots+k_n=k} a_{1j}^{k_1} \cdots a_{nj}^{k_n} \, x_1^{k_1} \cdots x_n^{k_n} \binom{k}{k_1,\dots,k_n} \frac{1}{k!}$$

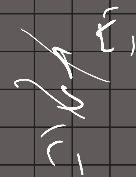
$$= \sum a_{ij}^{k_1} \cdots a_{nj}^{k_n} \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_n^{k_n}}{k_n!}$$

(3) Saw before: Action of SL_3 on sl_3



E_i, F_i



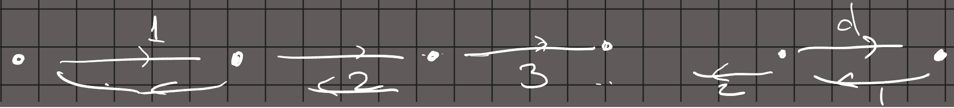


$E_1^{(2)} = E_1^2/2$ $\qquad \hat{E}_1^{(2)}$

$F_1^{(2)}$

(4) SL_2 standard reps $V^d = k[x,y]_d$

Chevalley basis: $x^d \quad \binom{d}{1} x^{d-1}y \quad \binom{d}{2} x^{d-2}y^2, \; \ldots \ldots \; y^d$



$d \qquad d-1 \qquad F^{(3)} v_\lambda$

$d \qquad d-2 \qquad -d$

$X, R, R^\vee \subset X^*$ $\qquad \xleftarrow[E]{} \qquad \xrightarrow[F]{} \qquad x\frac{\partial}{\partial y}, \quad y\,\partial/\partial x$

$\downarrow$

What is $\mathcal{O}_{\mathbb{Z}}(G)$?

Two ways: (1) (a) Start with char $k=0$, have standard V^λ's (irreducible). $\to$ Chevalley $\mathbb{Z}$ forms $V^\lambda_{\mathbb{Z}}$

(b) $\mathcal{O}_{\mathbb{Z}}(G)$ = subring (over $\mathbb{Z}$) of $\mathcal{O}_k(G)$ generated by matrix coefficients in Chevalley bases of all $V^\lambda_{\mathbb{Z}}$

$v_\lambda \otimes v_\mu$

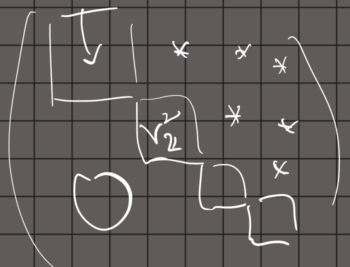
$V^\lambda_{\mathbb{Z}} \otimes V^\mu_{\mathbb{Z}}$ has a filtration,

$\uparrow$

$V^{\lambda+\mu}_{\mathbb{Z}}$ with $V^{\lambda+\mu}_{\mathbb{Z}}$ as a submodule,

and each quotient is some $V^\nu_{\mathbb{Z}}$.

$V^{\lambda+\mu}_{\mathbb{Z}}$



$F_i \mapsto F_i \otimes 1 + 1 \otimes F_i$

$F_i^m \mapsto \sum \binom{m}{k} F_i^k \otimes F_i^{m-k}$

$F_i^{(m)} \mapsto \sum F_i^{(k)} \otimes F_i^{(m-k)}$

(2) Define $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ $\qquad \mathcal{O}(G)^* \supseteq \mathcal{U}(\mathfrak{g})$

$\mathcal{U}(\mathfrak{g}) = \{ \xi \mid \xi$ vanishes on some $\mathfrak{m}_e^N \}$

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{t}) \otimes \mathcal{U}(\mathfrak{n}_+)$$

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{t} \oplus \mathfrak{n}_-$$

$\mathfrak{n}_+ \leftarrow K \cdot R_+$, $\mathfrak{t} = \mathrm{Lie}(T)$, $\mathfrak{n}_- \leftarrow K \cdot R_-$

$$\mathcal{U}_{\mathbb{Z}}(\mathfrak{g}) = \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-) \otimes \mathcal{U}_{\mathbb{Z}}(\mathfrak{t}) \otimes \mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)$$

$\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_-)$: $\mathbb{Z}$ subalg. of $\mathcal{U}_K(\mathfrak{n}_-)$ gen. by the $F_i^{(m)}$

$\mathcal{U}_{\mathbb{Z}}(\mathfrak{n}_+)$: $\mathbb{Z}$ subalg. of $\mathcal{U}_K(\mathfrak{n}_+)$ gen by the $E_i^{(m)}$

$T \qquad \mathcal{O}_K(T) = K(x_1^{\pm 1}, \dots, x_r^{\pm 1}]$

$\mathfrak{t} = K^r = K \cdot \{x_1 \frac{\partial}{\partial x_1}, \dots, x_r \frac{\partial}{\partial x_r}\}$

$x_1^{\lambda_1} \cdots x_r^{\lambda_r}$ are the characters of T

$T \to K^\times = \mathbb{G}_m$

$\mathcal{U}(\mathfrak{t}) = S(\mathfrak{t}) = K[h_1, \dots, h_r]$

$\mathfrak{t}$ acts on $V^\lambda = K \circlearrowleft T$

$x^\lambda \leftrightarrow \underline{x}$

$\mathcal{O}_{\mathbb{Z}}(T) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$

$x_1 \frac{\partial}{\partial x_1}(x^\lambda) = \lambda_1 \cdot x^\lambda$

<u>Prop.</u> $U_{\mathbb{Z}}(n_-) \otimes U_{\mathbb{Z}}(\mathfrak{t}) \otimes U_{\mathbb{Z}}(n_+)$ is a $\mathbb{Z}$-subalgebra of $\mathcal{U}(\mathfrak{g})$

This is $U_{\mathbb{Z}}(\mathfrak{g})$

Define $\mathcal{O}_{\mathbb{Z}}(G) = \{ f \in \mathcal{O}(G) \mid \langle f, \xi \rangle \in \mathbb{Z} \quad \forall\, \xi \in U_{\mathbb{Z}}(\mathfrak{g}) \}$

Same as $\mathcal{O}_{\mathbb{Z}}(G)$ generated matrix coeffs of the $V^{\lambda}_{\mathbb{Z}}$.

$\mathcal{U}_{\mathbb{Z}}(t)$ should be the $\mathbb{Z}$ dual to $(\mathbb{R}^\times)^r$ $T \cong (\mathbb{G}_m)^r$

$\mathcal{O}_{\mathbb{Z}}(T)$ inside $\mathcal{U}(t) = k[h_1, \dots, h_r]$

$$\mathcal{O}_k(T) \otimes \mathcal{U}_k(t) \longrightarrow k$$
$$f \otimes \xi \mapsto \xi(f)(e) \qquad \langle x^\lambda, \xi(h_1, \dots h_r) \rangle = \xi(\lambda_1, \dots, \lambda_r)$$

$$\mathcal{O}_{\mathbb{Z}}(T) \otimes \mathcal{U}_{\mathbb{Z}}(t) \longrightarrow \mathbb{Z}$$

Want all polynomials $\xi(h_1, \dots, h_r)$ s.t. $\xi \,|\, \mathbb{Z}^r$ is $\mathbb{Z}$-valued.

$\xi(h)$ has $\mathbb{Z}$ values for $h \in \mathbb{Z}$

$\iff \xi \in \mathbb{Z} \cdot \{ \binom{h}{m} \mid m \geq 0 \}$ ($\underline{\text{Thm}}$)

$$\binom{h}{m} = \frac{h(h-1)\cdots(h-m+1)}{m!} \in k[h]$$

$\mathcal{U}_{\mathbb{Z}}(t)$ has basis $\{ \binom{h_1}{m_1} \cdots \binom{h_r}{m_r} \}$